A short note on the 1st Chern class of a line bundle

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Abstract

Notes for 2020 BICMR Summer School for Differential Geometry.

1 Review of two definitions

Let X be a complex manifold, using the short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathscr{O} \xrightarrow{\exp(2\pi\sqrt{-1}\cdot)} \mathscr{O}^* \to 1$$

we get the exact sequence

$$\cdots \to H^1(X, \mathscr{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to \ldots .$$

We call $\delta : H^1(X, \mathscr{O}^*) \to H^2(X, \mathbb{Z})$ the "first Chern class" map.

Instead of holomorphic line bundles, we can consider C^∞ line bundles. These bundles are classified by $H^1(X, \mathcal{E}^*)$. Similarly, we have short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{E} \xrightarrow{\exp(2\pi\sqrt{-1}\cdot)} \mathcal{E}^* \to 1,$$

and consequently a short exact sequence:

$$\cdots \to H^1(X, \mathcal{E}) \cdots \to H^1(X, \mathcal{E}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{E}) \to \ldots .$$

Since $\mathcal{E}$ is a fine sheaf, we have $H^p(X, \mathcal{E}) = 0$ whenever $p \geq 1$. So $\delta : H^1(X, \mathcal{E}^*) \to H^2(X, \mathbb{Z})$ is an isomorphism (also called "first Chern class map"). This means that *complex line bundles are determined up to C^∞ isomorphisms by their first Chern class.*

On the other hand, we can use a connection on a given C^∞ complex line bundle L, and use the curvature form Θ to define

$$c_1(L) := \Big[\frac{\sqrt{-1}}{2\pi}\Theta\Big] \in H^2_{dR}(X; \mathbb{R}) \cong H^2(X, \mathbb{R}).$$

2 Relation between these two definitions

Since we have a natural homomorphism $\Phi : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$ using the sheaf morphism $\underline{\mathbb{Z}} \to \underline{\mathbb{R}}$. We shall explore the relation between $\Phi(\delta([L])) \in H^2(X,\mathbb{R})$ and $c_1(L) \in H^2_{dR}(X,\mathbb{R})$.

For simplicity, in the following we assume L is a holomorphic line bundle with Hermitian metric h. We leave the necessary modification in the general complex line bundle case as an exercise. (hint: you need to replace the Chern connection by any connection on the bundle, use the transformation formula for connection 1-forms when you change a frame.)

First recall the construction of $\delta : H^1(X,\mathscr{O}^*) \to H^2(X,\mathbb{Z})$. Let L be a complex line bundle. We use sufficiently fine locally finite trivializations $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ such that each U_α is simply connected and $H^*(X,\mathscr{O}^*)$ is isomorphic to $H^*(\mathcal{U},\mathscr{O}^*)$. Then $[L] \in H^1(X,\mathscr{O}^*)$ is determined by the Čech cocycle $\{\psi_{\alpha\beta}\}$, $\psi_{\alpha\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta)$. We define $\phi_{\alpha\beta} := \frac{1}{2\pi\sqrt{-1}}\log\psi_{\alpha\beta}$. Note that this is not a well-defined Čech cochain: *log* is a multi-valued function!

However, since $\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$, we get

$$z_{\alpha\beta\gamma} := \phi_{\alpha\beta} + \phi_{\beta\gamma} - \phi_{\alpha\gamma} \in \underline{\mathbb{Z}}(U_\alpha \cap U_\beta \cap U_\gamma).$$

This defines a Čech cocycle, whose cohomology class defines $\delta([L])$. Then $\Phi(\delta([L]))$ is also defined by $\{z_{\alpha\beta\gamma}\}$, just viewing $\underline{\mathbb{Z}}$ as a subsheaf of $\underline{\mathbb{R}}$.

To compare it with $c_1(L)$, we need a closer look at the de Rham isomorphism. We first break the resolution

$$0 \to \underline{\mathbb{R}} \to \mathscr{A}^0 \to \mathscr{A}^1 \to \dots$$

into short exact sequences:

$$0 \to \underline{\mathbb{R}} \to \mathscr{A}^0 \to \mathcal{K}_1 \to 0, \quad 0 \to \mathcal{K}_1 \to \mathscr{A}^1 \to \mathcal{K}_2 \to 0, \quad \dots$$

where $\mathcal{K}_i$ is the sheaf of closed i-forms. We get exact sequence for cohomology:

$$0 \to H^1(X,\mathcal{K}_1) \to H^2(X,\mathbb{R}) \to 0, \quad A^1(X) \to \mathcal{K}_2(X) \to H^1(X,\mathcal{K}_1) \to 0.$$

The first one gives $\delta_2 : H^1(X,\mathcal{K}_1) \cong H^2(X,\mathbb{R})$ and the second gives $\delta_1 : H^2_{dR}(X) \cong H^1(X,\mathcal{K}_1)$.

First we study δ_1: Our de Rham class is given by $\frac{\sqrt{-1}}{2\pi}\Theta(h) \in \mathcal{K}_2(X)$. Locally, we have $\Theta = d\theta_\alpha$, where $\theta_\alpha = \partial\log h_\alpha$, $h_\alpha = h(e_\alpha, e_\alpha)$, $e_\alpha(p) = \varphi_\alpha^{-1}(p,1)$. Then $\delta_1\Big([\frac{\sqrt{-1}}{2\pi}\Theta(h)]\Big)$ is given by $[\{\frac{\sqrt{-1}}{2\pi}(\theta_\beta - \theta_\alpha)\}]$.

Now

$$e_\beta(p) = \varphi_\beta^{-1}(p,1) = \varphi_\alpha^{-1} \circ (\varphi_\alpha \circ \varphi_\beta^{-1})(p,1) = \varphi_\alpha^{-1}(p,\psi_{\alpha\beta}(p)) = \psi_{\alpha\beta}(p)e_\alpha(p).$$

So we get $h_\beta = h_\alpha|\psi_{\alpha\beta}|^2$, and hence $\log h_\beta = \log h_\alpha + \log|\psi_{\alpha\beta}|^2$. So on $U_\alpha \cap U_\beta$, we have

$$\frac{\sqrt{-1}}{2\pi}(\theta_\beta - \theta_\alpha) = \frac{\sqrt{-1}}{2\pi}\partial \log|\psi_{\alpha\beta}|^2 = \frac{\sqrt{-1}}{2\pi}\partial \log \psi_{\alpha\beta} = \frac{\sqrt{-1}}{2\pi} d \log \psi_{\alpha\beta}.$$

Then $\delta_2\big([\{\frac{\sqrt{-1}}{2\pi}(\theta_\beta - \theta_\alpha)\}]\big)$ is represented by

$$\Big\{\frac{\sqrt{-1}}{2\pi}\Big(\log \psi_{\beta\gamma} - \log \psi_{\alpha\gamma} + \log \psi_{\alpha\beta}\Big)\Big\}.$$

This is precisely our $\{z_{\alpha\beta\gamma}\}$.

References

[1] Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, Inc., 1978.

[2] R.O. Wells, *Differential analysis on complex manifolds*, 2nd edition, Springer, 1980.